

# A Quantum Analogue of the $\mathcal{Z}$ Algebra

A. Hamid Bougourzi<sup>1</sup> and Luc Vinet<sup>2</sup>

*Centre de Recherches Mathématiques*

*Université de Montréal*

*C.P. 6128-A, Montréal, P.Q., H3C 3J7, Canada.*

## Abstract

We define a natural quantum analogue for the  $\mathcal{Z}$  algebra, and which we refer to as the  $\mathcal{Z}_q$  algebra, by modding out the Heisenberg algebra from the quantum affine algebra  $U_q(\widehat{sl(2)})$  with level  $k$ . We discuss the representation theory of this  $\mathcal{Z}_q$  algebra. In particular, we exhibit its reduction to a group algebra, and to a tensor product of a group algebra with a quantum Clifford algebra when  $k = 1$ , and  $k = 2$ , and thus, we recover the explicit constructions of  $U_q(\widehat{sl(2)})$ -standard modules as achieved by Frenkel-Jing and Bernard, respectively. Moreover, for arbitrary nonzero level  $k$ , we show that the explicit basis for the simplest  $\mathcal{Z}$ -generalized Verma module as constructed by Lepowsky and primc is also a basis for its corresponding  $\mathcal{Z}_q$ -module, i.e., it is invariant under the  $q$ -deformation for generic  $q$ . We expect this  $\mathcal{Z}_q$  algebra (associated with  $U_q(\widehat{sl(2)})$  at level  $k$ ), to play the role of a dynamical symmetry in the off-critical  $Z_k$  statistical models.

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<sup>1</sup> Email: bougourz@ere.umontreal.ca

<sup>2</sup> Email: vinet@ere.umontreal.ca

# 1 Introduction

One of the major recent developments in the field of integrable models has been the realization by the Kyoto school [1, 2] of the important role played by non-Abelian and dynamical symmetries in the resolution of integrable systems. Prior to this and besides conformal field theory, the main approach in the analysis of integrable models has been based on Abelian symmetries together with the Bethe ansatz. However, this approach, despite its success of being more systematic in handling the spectra of most integrable systems, has its limitations as far as concrete computations of physical quantities are concerned such as correlation functions and form factors. The reason is that the latter quantities are based on scalar products of the eigenvectors of the Hamiltonians or transfer matrices of the systems; but the eigenspaces in the physically interesting thermodynamic limit are infinite-dimensional, and hence it is not easy to define their structures, and much less the scalar products on them. However, since some non-Abelian infinite-dimensional algebras have well defined scalar products on their infinite-dimensional modules, then if we succeed in establishing that the Hamiltonian or the transfer matrix of an integrable system commutes with one of these algebras, we automatically know, not only its eigenspaces which are the modules of this algebra, but also the scalar product on them. We might even describe the local operators, and the creation and annihilation operators of the eigenvectors in terms of some operators related to this algebra, such as the intertwiners of its modules, and hence we might be able to compute exactly the correlation functions and the form factors. In fact, this is precisely the program that has been behind the enormous success in the resolution of conformal field theories, and more recently in the resolution of the XXZ quantum spin chain model (which is equivalent to the 6-vertex classical model) in the antiferromagnetic regime by the Kyoto school. We should however, mention Ref. [3] where another approach to the calculation of correlation functions is developed.

It is then an interesting program to build as many infinite-dimensional algebras as possible, hoping that one of them might turn out to be a non-Abelian or a dynamical symmetry of an integrable model, and vice versa. The main point of this paper is precisely to define a new infinite-dimensional algebra, which, as explained in the next paragraph, should be the

dynamical symmetry of the off-critical  $Z_k$  models.

It is known that the  $\widehat{sl(2)}$  affine Lie algebra is a dynamical symmetry in conformal field theory (the continuum critical limit of the XXX model) [4] and a non-Abelian symmetry of the antiferromagnetic (off-critical) XXX model [2]. Moreover, its deformation, the  $U_q(\widehat{sl(2)})$  algebra, is also a non-Abelian symmetry of the antiferromagnetic (off critical) XXZ model. It is also known that the critical  $Z_k$  models, such as the Ising model ( $k = 2$ ) and the Potts model ( $k = 3$ ), have as a dynamical symmetry the parafermionic algebra [5], which is ironically related to the  $\mathcal{Z}$  algebra [6]. The latter algebra, in turn, is obtained from the quotient of the  $\widehat{sl(2)}$  algebra with level  $k$  by its Heisenberg subalgebra.

From all the above known results, it is therefore natural to consider a similar construction for a quantum analogue of the  $\mathcal{Z}$  algebra, denoted by  $\mathcal{Z}_q$ , from the  $U_q(\widehat{sl(2)})$  algebra, and to expect it to play the same role of a dynamical symmetry for the off-critical  $Z_k$  models. In fact, such a program is already and implicitly implemented in the simplest case of the off-critical Ising model ( $k = 2$ ), where the  $\mathcal{Z}_q$  algebra (to be precise the corresponding quantum parafermionic algebra) reduces simply to a quantum Clifford algebra [7].

This paper is organized as follows: in section 2, we recall basic definitions about the  $U_q(\widehat{sl(2)})$  quantum affine algebra, using the formal variable approach. In section 3, we gradually introduce the  $\mathcal{Z}_q$  algebra by modding out the Heisenberg subalgebra from  $U_q(\widehat{sl(2)})$  with level  $k$ . We derive two defining relations, called “the quantum generalized commutation relations”, for this algebra, as well as the relations between its elements and those of  $U_q(\widehat{sl(2)})$ . In section 4, we discuss the reduction of  $\mathcal{Z}_q$  in the simpler cases  $k = 1$ , and  $k = 2$  to a group algebra  $\mathbb{C}[Q]$  with  $Q$  being the root lattice of the  $sl(2)$  Lie algebra [8], and to a tensor product of  $\mathbb{C}[Q]$  with a quantum deformation of a Clifford algebra [9], respectively. Then, we provide an explicit construction of the basis of the simplest  $\mathcal{Z}_q$  modules, and hence  $U_q(\widehat{sl(2)})$  modules, for arbitrary non-zero level  $k$ . These are the so called generalized Verma modules [10, 11]. We show that the spanning vectors of the basis of a generalized Verma module as constructed in the ‘classical’  $\mathcal{Z}$  algebra case in [11, 12], do still form a basis for a  $\mathcal{Z}_q$ -generalized Verma module. In section 5, we give more quantum generalized commutation relations satisfied by polynomials of  $\mathcal{Z}_q$  elements. We think that these relations will eventually be useful in the explicit constructions of all the  $\mathcal{Z}_q$ -modules, and from them all the

$U_q(\widehat{sl(2)})$ -modules, and especially the standard ones with arbitrary non-zero level  $k$ . Finally, Section 6 is devoted to our conclusions.

## 2 The $U_q(\widehat{sl(2)})$ quantum affine algebra

The  $U_q(\widehat{sl(2)})$  affine algebra is a unital associative algebra with elements  $\{e_{\pm\alpha_i}, k_i^{\pm}, q^{\pm d}; i = 0, 1\}$  and defining relations in the homogeneous gradation [13, 14]

$$\begin{aligned}
[k_i, k_j] &= 0, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \\
q^{-d} q^d &= q^d q^{-d} = 1, \\
q^d k_i q^{-d} &= k_i, \quad q^d e_{\pm\alpha_i} q^{-d} = q^{\pm\delta_{i0}} e_{\pm\alpha_i}, \\
k_i e_{\pm\alpha_j} k_i^{-1} &= q^{\pm(\alpha_i, \alpha_j)} e_{\pm\alpha_j}, \\
[e_{\alpha_i}, e_{\alpha_j}] &= \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\
(e_{\pm\alpha_i})^3 e_{\pm\alpha_j} - [3](e_{\pm\alpha_i})^2 e_{\pm\alpha_j} e_{\pm\alpha_i} + [3]e_{\pm\alpha_i} e_{\pm\alpha_j} (e_{\pm\alpha_i})^2 - e_{\pm\alpha_j} (e_{\pm\alpha_i})^3 &= 0,
\end{aligned} \tag{2.1}$$

where  $[x] = (q^x - q^{-x})/(q - q^{-1})$ ,  $q = e^{t/2}$  is a complex number called a deformation parameter, and  $\{\alpha_i, i = 0, 1\}$  is the set of positive simple roots of  $\widehat{sl(2)}$  affine Lie algebra with the invariant symmetric bilinear form  $(\alpha_i, \alpha_j) = a_{i,j}$  (see Section 4). Here  $a_{i,i} = 2$  and  $a_{i,1-i} = -2$ ,  $i = 0, 1$  are the elements of the  $\widehat{sl(2)}$  affine Cartan matrix. Note that the special element  $\gamma = k_0 k_1$  is in the center of  $U_q(\widehat{sl(2)})$  and acts as  $q^k$  on its highest weight representations, with  $k$  referred to as the level. We also refer to the above set of elements generating  $U_q(\widehat{sl(2)})$  as the Chevalley basis.

The Chevalley basis consists of elements associated with the simple roots only. One would like to describe the commutation relations of all basis elements associated with the infinite-dimensional set of roots  $\{\pm\alpha + n\delta; n \in \mathbb{Z}\} \cup \{n\delta; n \in \mathbb{Z} \setminus \{0\}\}$ , with  $\alpha = \alpha_1$  and  $\delta = \alpha_0 + \alpha_1$ , and where the Serre relation (i.e., the last relation in (2.1)) becomes redundant. Drinfeld succeeded in finding such a basis, which we refer as the Drinfeld basis [15]. It is generated by the elements  $\{x_n^{\pm}, \alpha_m, K^{\pm}, q^{\pm d}, \gamma^{\pm 1/2}; n \in \mathbb{Z}, m \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}\}$  with defining relations

$$\gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1, \quad [\gamma^{\pm 1/2}, y] = 0, \quad \forall y \in U_q(\widehat{sl(2)}), \tag{2.2}$$

$$K K^{-1} = K^{-1} K = 1, \tag{2.3}$$

$$K\alpha_n K^{-1} = \alpha_n, \quad (2.4)$$

$$Kx_n^\pm K^{-1} = q^{\pm 2}x_n^\pm, \quad (2.5)$$

$$q^d q^{-d} = q^{-d} q^d = 1, \quad Kq^{\pm d} K^{-1} = q^{\pm d}, \quad (2.6)$$

$$q^d x_n^\pm q^{-d} = q^n x_n^\pm, \quad (2.7)$$

$$q^d \alpha_n q^{-d} = q^n \alpha_n, \quad (2.8)$$

$$[\alpha_n, \alpha_m] = \frac{(q^{2n} - q^{-2n})(\gamma^n - \gamma^{-n})}{nt^2} \delta_{n+m,0}, \quad (2.9)$$

$$[\alpha_n, x_m^\pm] = \pm \sqrt{2} \frac{\gamma^{\mp|n|/2} (q^{2n} - q^{-2n})}{2nt} x_{n+m}^\pm, \quad (2.10)$$

$$[x_n^+, x_m^-] = \frac{\gamma^{(n-m)/2} \Psi_{n+m} - \gamma^{(m-n)/2} \Phi_{n+m}}{q - q^{-1}}, \quad (2.11)$$

$$x_{n+1}^\pm x_m^\pm - q^{\pm 2} x_m^\pm x_{n+1}^\pm = q^{\pm 2} x_n^\pm x_{m+1}^\pm - x_{m+1}^\pm x_n^\pm, \quad (2.12)$$

where  $\Psi_n$  and  $\Phi_n$  are given by the mode expansions of the fields  $\Psi(z)$  and  $\Phi(z)$ , which are themselves defined by

$$\begin{aligned} \Psi(z) &= \sum_{n \geq 0} \Psi_n z^{-n} = K \exp\{t \sum_{n > 0} \alpha_n z^{-n}\}, \\ \Phi(z) &= \sum_{n \leq 0} \Phi_n z^{-n} = K^{-1} \exp\{-t \sum_{n < 0} \alpha_n z^{-n}\}. \end{aligned} \quad (2.13)$$

Here  $z$  is a formal variable and

$$K = \Psi_0 = \Phi_0^{-1} \equiv q^{\alpha_0}, \quad (2.14)$$

where we mean identification by the symbol  $\equiv$ .

The isomorphism  $\rho$  between  $U_q(\widehat{sl(2)})$  in the Chevalley basis and  $U_q(\widehat{sl(2)})$  in the Drinfeld basis is given explicitly by

$$\begin{aligned} \rho : \quad k_0 &\rightarrow \gamma K^{-1}, \\ \rho : \quad k_1 &\rightarrow K, \\ \rho : \quad e_{\pm \alpha_1} &\rightarrow x_0^\pm, \\ \rho : \quad e_{\alpha_0} &\rightarrow x_1^- K^{-1}, \\ \rho : \quad e_{-\alpha_0} &\rightarrow K x_{-1}^+. \end{aligned} \quad (2.15)$$

For later purposes we will use the formal variable approach [16] (instead of the usual operator product expansion method) to re-express the algebra as a quantum current algebra with elements  $\{\Psi(z), \Phi(z), x^\pm(z), \gamma^{\pm 1/2}, q^{\pm d}\}$ , where [8]

$$x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n}, \quad (2.16)$$

and with defining relations

$$\gamma^{1/2}\gamma^{-1/2} = \gamma^{-1/2}\gamma^{1/2} = 1, \quad [\gamma^{\pm 1/2}, y] = 0, \quad \forall y \in U_q(\widehat{sl(2)}), \quad (2.17)$$

$$[\Psi(z), \Psi(w)] = 0, \quad (2.18)$$

$$[\Phi(z), \Phi(w)] = 0, \quad (2.19)$$

$$\Psi(z)\Phi(w) = g(wz^{-1}\gamma)g(wz^{-1}\gamma^{-1})^{-1}\Phi(w)\Psi(z), \quad (2.20)$$

$$\Psi(z)x^\epsilon(w) = g(wz^{-1}\gamma^{-\epsilon/2})^{-\epsilon}x^\epsilon(w)\Psi(z), \quad (2.21)$$

$$\Phi(z)x^\epsilon(w) = g(zw^{-1}\gamma^{-\epsilon/2})^\epsilon x^\epsilon(w)\Phi(z), \quad (2.22)$$

$$[x^\epsilon(z), x^{-\epsilon}(w)] = \epsilon \frac{\delta(zw^{-1}\gamma^{-\epsilon})\Psi(w\gamma^{\epsilon/2}) - \delta(zw^{-1}\gamma^\epsilon)\Phi(z\gamma^{\epsilon/2})}{q - q^{-1}}, \quad (2.23)$$

$$(z - wq^{2\epsilon})x^\epsilon(z)x^\epsilon(w) = (zq^{2\epsilon} - w)x^\epsilon(w)x^\epsilon(z), \quad (2.24)$$

$$q^d x^\epsilon(z) = x^\epsilon(zq^{-1})q^d, \quad (2.25)$$

$$q^d \Psi(z) = \Psi(zq^{-1})q^d, \quad (2.26)$$

$$q^d \Phi(z) = \Phi(zq^{-1})q^d. \quad (2.27)$$

Here  $\epsilon = \pm 1$  and  $g(z)$  is meant to be the following formal power series in  $z$ :

$$g(z) = \sum_{n \in \mathbb{Z}_+} c_n z^n, \quad (2.28)$$

where the coefficients  $c_n, n \in \mathbb{Z}_+$  are determined from the Taylor expansion of the function

$$f(\xi) = \frac{q^2 \xi - 1}{\xi - q^2} = \sum_{n \in \mathbb{Z}_+} c_n z^n \quad (2.29)$$

at  $\xi = 0$ . Note that for this reason  $f(\xi^{-1}) = f(\xi)^{-1}$  but  $g(z^{-1}) \neq g(z)^{-1}$ . In fact,  $g(z)^{-1}$  is obtained from  $f(\xi)^{-1}$  in the same manner as  $g(z)$  is obtained from  $f(\xi)$  [8]. In the above relations we have also introduced the famous  $\delta$ -function  $\delta(z)$  which is defined as the formal Laurent series

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n, \quad (2.30)$$

and which plays a key role in the formal calculus approach [16]. Its main properties are summarized by the following relations:

$$\begin{aligned} \delta(z) &= \delta(z^{-1}), \\ \delta(z) &= \frac{1}{z} + \frac{z^{-1}}{1-z^{-1}}, \\ G(z, w)\delta(azw^{-1}) &= G(z, az)\delta(azw^{-1}) = G(a^{-1}w, w)\delta(azw^{-1}), \quad a \in \mathbb{C}^*, \end{aligned} \quad (2.31)$$

where  $G(z, w)$  is any operator with a formal Laurent expansion in  $z$  and  $w$  given by

$$G(z, w) = \sum_{n, m \in \mathbb{Z}} G_{n, m} z^n w^m. \quad (2.32)$$

Note that it is crucial that both  $\delta(z)$  and  $G(z, w)$  have expansions in integral powers of  $z$  and  $w$ , otherwise the above properties of the  $\delta$ -function will not hold.

The three relations (2.25), (2.26) and (2.27) translate the fact that  $x_n^\pm$ ,  $\Psi_n$  and  $\Phi_n$  are homogeneous of the same degree  $n$ .

### 3 The $\mathcal{Z}_q$ algebra

It is well known that the Heisenberg subalgebra of  $\widehat{sl(2)}$  plays a crucial role in the construction of vertex operators and highest weight representations. One would like to extend this role to the quantum case. Note that in the sequel, the unit element and  $q^{\pm d}$  are meant to be in all the (sub)algebras defined below, so we will not consider them unless stated otherwise. Let  $U_q(\hat{h})$  be the quantum analogue of the enveloping Heisenberg algebra, referred to as  $q$ -Heisenberg algebra. It is a subalgebra of  $U_q(\widehat{sl(2)})$  generated by  $\{\alpha_n, \gamma^{\pm 1/2}, n \in \mathbb{Z}^*\}$  with relations:

$$[\alpha_n, \gamma^{\pm 1/2}] = 0, \quad \gamma^{1/2} \gamma^{-1/2} = \gamma^{-1/2} \gamma^{1/2} = 1, \quad (3.33)$$

$$[\alpha_n, \alpha_m] = \delta_{n+m, 0} \frac{(q^{2n} - q^{-2n})(\gamma^n - \gamma^{-n})}{2nt^2}. \quad (3.34)$$

Let  $U_q(\hat{h}^+)$  and  $U_q(\hat{h}^-)$  denote the commutative subalgebras of  $U_q(\hat{h})$  generated by  $\{\alpha_n, \gamma^{\pm 1/2}; n > 0\}$  and  $\{\alpha_n; n < 0\}$  respectively. By the quantum analogue of Poincaré-Birkhoff-Witt theorem for  $U_q(\hat{h})$ , we have

$$U_q(\hat{h}) = U_q(\hat{h}^+) U_q(\hat{h}^-), \quad (3.35)$$

and consequently, the following induced  $\hat{h}$ -module:

$$I(q^k) = U_q(\hat{h}) \otimes_{U_q(\hat{h}^+)} \mathbb{C}[q^k] \quad (3.36)$$

is irreducible and isomorphic to  $U_q(\hat{h}^-)$  and hence to the symmetric algebra  $S(\hat{h}^-)$  [8]. In this formula,  $\mathbb{C}[q^k]$  denotes the field of complex numbers considered as the one-dimensional  $U_q(\hat{h}^+)$ -module and on which  $\gamma$  acts as multiplication by  $q^k$  and  $\alpha_n, n > 0$  acts trivially. This

means that  $S(\hat{h}^-)$  is a canonical  $U_q(\hat{h})$ -module on which  $\gamma$  acts as multiplication by  $q^k$ ,  $\alpha(n)$  ( $n < 0$ ) acts as a creation (multiplication) operator, and  $\alpha(n)$  ( $n > 0$ ) acts as an (derivation) annihilation operator satisfying the relation (3.34). Henceforth,  $\alpha(n)$  denotes the generator  $\alpha_n$  on  $S(\hat{h}^-)$ . More precisely, the latter actions are given by:

$$\begin{aligned}\gamma^{\pm 1} : \quad x &\rightarrow q^{\pm k}x, \\ \alpha(n) : \quad x &\rightarrow \alpha_n x, \quad n < 0, \\ \alpha(n) : \quad x &\rightarrow [\alpha_n, x], \quad n > 0,\end{aligned}\tag{3.37}$$

where  $x$  is any element in  $S(\hat{h}^-)$ . Moreover, the action of  $q^{\pm d}$  on  $S(\hat{h}^-)$  is defined by

$$q^{\pm d} : \quad x \rightarrow q^{\pm d} x q^{\pm d}.\tag{3.38}$$

We also denote by  $\Psi(n)$  and  $\Phi(n)$  the generators  $\Psi_n$  and  $\Phi_n$  on  $S(\hat{h}^-)$  respectively, that is,  $\Psi(n)$  and  $\Phi(n)$  are related to  $\alpha(n)$  in the same way as  $\Psi_n$  and  $\Phi_n$  are related to  $\alpha_n$  (see (2.13). The action of  $\alpha(0)$  and hence the actions of  $\Psi(0) = q^{\alpha(0)}$  and  $\Phi(0) = q^{-\alpha(0)}$  on  $S(\hat{h}^-)$  will be defined later.

Now we would like to show that the highest weight modules of the whole quantum affine algebra  $U_q(\widehat{sl(2)})$  must be constructed as tensor products of the form  $S(\hat{h}^-) \otimes W$ . Here  $W$  are certain vector spaces to be defined later and which are trivial as  $U_q(\hat{h})$ -module. *La raison d'être* of  $W$  stems from the fact that  $S(\hat{h}^-)$  is only a  $U_q(\hat{h})$ -module and in general cannot be upgraded to a  $U_q(\widehat{sl(2)})$ -module, which is especially true here since we are considering  $U_q(\widehat{sl(2)})$  in the homogeneous gradation. This is because it is well known that the  $\widehat{sl(2)}$ -highest weight modules in the homogeneous gradation are constructed as the above tensor products with non-trivial  $W$  spaces [11], and so we expect this to be also true for the  $U_q(\widehat{sl(2)})$ -highest weight modules. Therefore, we have to “correct”  $S(\hat{h}^-)$  by tensoring it with additional new spaces which do not overlap with it, so that the resulting tensor product remains as a  $U_q(\hat{h})$ -module. Of course, this correction will be performed by considering a minimum number of extra spaces.

It is well known in the case of affine algebras that these constructions can be achieved by means of vertex operators. The most famous vertex construction of the quantum affine algebras is the Frenkel-Jing one, which is however valid only for the simply laced algebras and with the central element  $\gamma$  acting as  $q$  (i.e.,  $k = 1$ ) on their highest weight modules



[8]. In this case, which will be recovered explicitly later for  $U_q(\widehat{sl(2)})$  when we set  $k = 1$  in our general construction, it turns out that the required minimal correction consists in tensoring the symmetric algebra with a group algebra associated with the root lattice of the Lie algebra corresponding to the quantum affine algebra in question. However, if  $k > 1$  the latter correction is not sufficient in the sense that it requires extra spaces. For example, if  $k = 2$  one needs to consider, in addition to the group and symmetric algebras, an exterior (Clifford) algebra. Such a construction has been achieved in the case of  $U_q(so(2n+1))$  with level 1 by Bernard [9]. For  $k > 2$  it was shown in [17] that one needs to introduce besides the group and symmetric algebras, a certain quantum parafermionic algebra (though the representation theory was not discussed there). In fact, when  $k > 2$  we find it easier for our purposes here to discuss the representation theory of the whole  $\mathcal{Z}_q$  algebra rather than its decomposition as a tensor product of a group algebra and a parafermionic algebra.

Although we are concerned with  $U_q(\widehat{sl(2)})$  for  $k > 1$ , the form of the vertex operators used by Frenkel and Jing for  $k = 1$  led us to introduce the following vertex operators

$$S_\epsilon^\pm(z) = \exp\left\{\pm\epsilon t \sum_{n>0} \frac{\alpha(\pm n)}{q^{nk} - q^{-nk}} q^{-\epsilon nk/2} z^{\mp n}\right\}, \quad \epsilon = \pm, \quad (3.39)$$

which are viewed as formal Laurent series in  $z$  with coefficients acting on  $S(\hat{h}^-)$ . Strictly speaking, when  $k = 1$  the above vertex operators reduce to the inverse operators of those considered by Frenkel and Jing. Using (3.34) and the usual formal rule

$$e^A e^B = e^B e^A e^{[A,B]} \quad \text{if} \quad [A, [A, B]] = [B, [A, B]] = 0, \quad (3.40)$$

for some operators  $A$  and  $B$ , we find

$$\begin{aligned} S_\epsilon^+(z) S_{\epsilon'}^-(w) &= \frac{(q^{k-2-(\epsilon+\epsilon')k/2} w z^{-1}; q^{2k})_\infty^{\epsilon\epsilon'}}{(q^{k+2-(\epsilon+\epsilon')k/2} w z^{-1}; q^{2k})_\infty^{\epsilon\epsilon'}} S_{\epsilon'}^-(w) S_\epsilon^+(z), \\ S_\epsilon^\pm(z) S_{\epsilon'}^\pm(w) &= S_{\epsilon'}^\pm(w) S_\epsilon^\pm(z), \end{aligned} \quad (3.41)$$

where as usual  $(x; y)_\infty$  means

$$(x; y)_\infty = \prod_{n=0}^{\infty} (1 - xy^n). \quad (3.42)$$

Each factor  $(1 - wz^{-1}q^x)^{-1}$  in (3.41) is understood as the formal power series  $\sum_{n \geq 0} w^n z^{-n} q^{xn}$ .

Moreover, using (3.34) and the formal rule

$$[A, e^B] = [A, B] e^B, \quad \text{if} \quad [B, [A, B]] = 0, \quad (3.43)$$

for some operators  $A$  and  $B$ , we obtain

$$\begin{aligned}
[\alpha(n), S_\epsilon^+(z)] &= 0, \\
[\alpha(-n), S_\epsilon^+(z)] &= -\frac{\epsilon q^{-\epsilon nk/2} z^{-n} (q^{2n} - q^{-2n})}{nt} S_\epsilon^+(z), \\
[\alpha(-n), S_\epsilon^-(z)] &= 0, \\
[\alpha(n), S_\epsilon^-(z)] &= -\frac{\epsilon q^{-\epsilon nk/2} z^n (q^{2n} - q^{-2n})}{nt} S_\epsilon^-(z),
\end{aligned} \tag{3.44}$$

where  $n > 0$  and  $\epsilon = \pm$ . For future purposes let us note here that one can easily show that the commutation relations (2.21) and (2.22) are equivalent to

$$[\alpha_n, x^\epsilon(z)] = \frac{\epsilon q^{-\epsilon |n|k/2} z^n (q^{2n} - q^{-2n})}{nt} x^\epsilon(z), \quad \epsilon = \pm; \quad n \in \mathbb{Z} \setminus \{0\}. \tag{3.45}$$

Let us now define the main new objects of this paper, which we refer to as the “ $\mathcal{Z}_q$  operators  $\mathcal{Z}_n^\epsilon$ ,” as the Laurent modes in

$$\mathcal{Z}^\epsilon(z) = \sum_{n \in \mathbb{Z}} \mathcal{Z}_n^\epsilon z^{-n}, \tag{3.46}$$

where

$$\mathcal{Z}^\epsilon(z) = S_\epsilon^-(z) x^\epsilon(z) S_\epsilon^+(z). \tag{3.47}$$

These operators  $\mathcal{Z}_n^\epsilon$  are the quantum analogues of the (classical,  $q = 1$ )  $\mathcal{Z}$ -operators that have been extensively studied in the literature (see for example Reference [12]). By abuse of terminology, we refer also to the “currents  $\mathcal{Z}^\epsilon(z)$ ” as  $\mathcal{Z}_q$  operators, but strictly speaking they are the generating functions of the latter operators. Let us stress here that  $z$  is a formal variable,  $\mathbb{Z}$  is the set of integer numbers, whereas  $\mathcal{Z}_n^\epsilon$  are operators so there should not be any confusion with these notations. Let us also denote by  $\mathcal{Z}_q$  the algebra generated by  $\{\Psi_0, \Phi_0, \mathcal{Z}_n^\epsilon; n \in \mathbb{Z}\}$ . The defining relations of this algebra, which we refer to as “the quantum generalized commutation relations” will be given shortly below.

The space  $W$  on which this algebra acts non-trivially is the necessary space to be tensored with  $S(\hat{h}^-)$  such that  $S(\hat{h}^-) \otimes W$  is a  $U_q(\widehat{sl(2)})$ -module. Therefore  $W$  will be defined if we know all the properties of the  $\mathcal{Z}_q$  operators, that is, their relations with  $U_q(\widehat{sl(2)})$  itself and their algebra (i.e., quantum generalized commutation relations).

For this purpose, let  $\mathcal{Z}(\epsilon|n)$  and  $X(\epsilon|n)$  represent the actions of  $\mathcal{Z}_n^\epsilon$  and  $x_n^\epsilon$  on  $W$  and  $S(\hat{h}^-) \otimes W$  respectively. By definition, the relation between the latter operators is given through their generating functions by (3.47). Next, it can easily be checked that the relations

(3.44) and (3.45) imply that the  $\mathcal{Z}_q$  operators commute with the quantum Heisenberg algebra  $U_q(\hat{h})$ , i.e.,

$$\begin{aligned} [\alpha(n), \mathcal{Z}(\epsilon|z)] &= 0, \quad n \in \mathbb{Z} \setminus \{0\}, \\ [\gamma^{\pm 1/2}, \mathcal{Z}(\epsilon|z)] &= 0, \quad \epsilon = \pm. \end{aligned} \quad (3.48)$$

This is a very important result, which is in fact the main motivation behind the particular choice for the forms of  $S_\epsilon^\pm(z)$  as given by (3.39). This is because the symmetric algebra  $S(\hat{h}^-)$  realizes already the quantum Heisenberg subalgebra  $U_q(\hat{h})$  and since the  $\mathcal{Z}_q$  operators commute with  $U_q(\hat{h})$ , we can define then the actions of  $U_q(\hat{h})$  and  $\mathcal{Z}_q$  as follows on  $S(\hat{h}^-) \otimes W$ :

$$\begin{aligned} x : u \otimes v &\rightarrow xu \otimes v, \\ y : u \otimes v &\rightarrow u \otimes yv, \end{aligned} \quad (3.49)$$

where  $x \in U_q(\hat{h})$ ,  $y \in \mathcal{Z}_q$ ,  $u \in S(\hat{h}^-)$ , and  $v \in W$ . From the Laurent expansion in  $zw^{-1}$  of both sides of (2.21) and (2.22), we obtain the relations

$$\begin{aligned} \Psi(0)X(\epsilon|w) &= q^{2\epsilon}X(\epsilon|w)\Psi(0), \\ \Phi(0)X(\epsilon|w) &= q^{-2\epsilon}X(\epsilon|w)\Phi(0), \end{aligned} \quad (3.50)$$

which, because of the relation (2.5), amount to

$$\begin{aligned} \Psi(0)\mathcal{Z}(\epsilon|w) &= q^{2\epsilon}\mathcal{Z}(\epsilon|w)\Psi(0), \\ \Phi(0)\mathcal{Z}(\epsilon|w) &= q^{-2\epsilon}\mathcal{Z}(\epsilon|w)\Phi(0). \end{aligned} \quad (3.51)$$

Combining these relations with (3.48) and

$$\begin{aligned} \Psi(z) &= \Psi_0 S_\epsilon^+(zq^{-3\epsilon k/2}) S_{-\epsilon}^+(zq^{3\epsilon k/2}), \quad \epsilon = \pm, \\ \Phi(z) &= \Phi_0 S_\epsilon^-(zq^{3\epsilon k/2}) S_{-\epsilon}^-(zq^{-3\epsilon k/2}), \quad \epsilon = \pm, \end{aligned} \quad (3.52)$$

we arrive finally at

$$\begin{aligned} \Psi(n)\mathcal{Z}(\epsilon|w) &= q^{2\epsilon}\mathcal{Z}(\epsilon|w)\Psi(n), \quad n \geq 0, \\ \Phi(n)\mathcal{Z}(\epsilon|w) &= q^{-2\epsilon}\mathcal{Z}(\epsilon|w)\Phi(n), \quad n \leq 0. \end{aligned} \quad (3.53)$$

In the sequel, however, we will only consider the relations (3.48) and (3.51) but not (3.53) since the latter is an immediate consequence of the former and (3.52).

Clearly, the action of the generating functions

$$X(\epsilon|z) = \sum_{n \in \mathbb{Z}} X(\epsilon|n) z^{-n} \quad (3.54)$$

on  $S(\hat{h}^-) \otimes W \otimes \mathbb{C}[z, z^{-1}]$  decomposes then as

$$X(\epsilon|z) = S_{-\epsilon}^-(zq^{-\epsilon k})S_{-\epsilon}^+(zq^{\epsilon k}) \otimes \mathcal{Z}(\epsilon|z), \quad \epsilon = \pm, \quad (3.55)$$

where we have used

$$(S_{\epsilon}^{\pm}(z))^{-1} = S_{-\epsilon}^{\pm}(zq^{\pm \epsilon k}), \quad \epsilon = \pm, \quad (3.56)$$

and (3.44) to express  $X_{\epsilon}(z)$  in terms of  $\mathcal{Z}(\epsilon|z)$ . We now define the actions of  $q^{\pm d}$  and  $\gamma^{\pm 1}$  on  $S(\hat{h}^-) \otimes W$  as

$$\begin{aligned} q^{\pm d} : u \otimes v &\rightarrow q^{\pm d}u \otimes q^{\pm d}v, \\ \gamma^{\pm 1} : u \otimes v &\rightarrow q^{\pm k}u \otimes v, \end{aligned} \quad (3.57)$$

where  $u \in S(\hat{h}^-)$  and  $v \in W$ . The relation between  $\mathcal{Z}(\epsilon|z)$  and  $q^{\pm d}$ , which reads as

$$q^d \mathcal{Z}(\epsilon|z) = \mathcal{Z}(\epsilon|zq^{-1})q^d, \quad (3.58)$$

can easily be derived from (2.25) and

$$q^d S_{\epsilon'}^{\epsilon}(z) = S_{\epsilon'}^{\epsilon}(zq^{-1})q^d, \quad (3.59)$$

which, in turn, can be obtained from (3.40). Relation (3.58) means that the  $\mathcal{Z}_q$  algebra is graded (it inherits the gradation of  $U_q(\widehat{sl(2)})$ ) and that the  $\mathcal{Z}_q$  operators  $\mathcal{Z}(\epsilon|n)$  are homogeneous of degree  $n$ .

Let us now turn to the derivation of the defining relations (besides (2.14) and (3.51)) of the  $\mathcal{Z}_q$  algebra, that is, the quantum generalized commutation relations. They simply follow from the substitution of  $X(\epsilon|z)$  as given by (3.55) in (2.23) and (2.24). We find

$$\begin{aligned} & \frac{(q^{k+2}wz^{-1}; q^{2k})_{\infty}}{(q^{k-2}wz^{-1}; q^{2k})_{\infty}} \mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|w) - \frac{(q^{k+2}zw^{-1}; q^{2k})_{\infty}}{(q^{k-2}zw^{-1}; q^{2k})_{\infty}} \mathcal{Z}(-\epsilon|w) \mathcal{Z}(\epsilon|z) \\ &= \frac{\epsilon}{q - q^{-1}} \left( \Psi_0 \delta(zw^{-1}q^{-\epsilon k}) - \Phi_0 \delta(zw^{-1}q^{\epsilon k}) \right) \\ &= \frac{1}{q - q^{-1}} \left( q^{\epsilon \alpha(0)} \delta(zw^{-1}q^{-k}) - q^{-\epsilon \alpha(0)} \delta(zw^{-1}q^k) \right), \end{aligned} \quad (3.60)$$

$$\begin{aligned} & (z - q^{2\epsilon}w) \frac{(q^{k-2-k\epsilon}wz^{-1}; q^{2k})_{\infty}}{(q^{k+2-k\epsilon}wz^{-1}; q^{2k})_{\infty}} \mathcal{Z}(\epsilon|z) \mathcal{Z}(\epsilon|w) \\ &= (q^{2\epsilon}z - w) \frac{(q^{k-2-k\epsilon}zw^{-1}; q^{2k})_{\infty}}{(q^{k+2-k\epsilon}zw^{-1}; q^{2k})_{\infty}} \mathcal{Z}(\epsilon|w) \mathcal{Z}(\epsilon|z). \end{aligned} \quad (3.61)$$

In summary, in addition to the latter quantum generalized commutation relations, the  $\mathcal{Z}_q$  fields  $\mathcal{Z}(\epsilon|z)$  must satisfy the following relations with the elements of  $U_q(\widehat{sl(2)})$  when they act on the  $W$  part of the tensor product  $S(\hat{h}^-) \otimes W$ :

$$[\alpha(n), \mathcal{Z}(\epsilon|z)] = 0, \quad n \in \mathbb{Z} \setminus \{0\}, \quad (3.62)$$

$$\Psi(0)\mathcal{Z}(\epsilon|z) = q^{2\epsilon}\mathcal{Z}(\epsilon|z)\Psi(0), \quad (3.63)$$

$$\Phi(0)\mathcal{Z}(\epsilon|z) = q^{-2\epsilon}\mathcal{Z}(\epsilon|z)\Phi(0), \quad (3.64)$$

$$X(\epsilon|z) = S_{-\epsilon}^-(zq^{-\epsilon k})S_{-\epsilon}^+(zq^{\epsilon k}) \otimes \mathcal{Z}(\epsilon|z), \quad (3.65)$$

$$q^d\mathcal{Z}(\epsilon|z) = \mathcal{Z}(\epsilon|zq^{-1})q^d, \quad (3.66)$$

$$[\gamma^\pm, \mathcal{Z}(\epsilon|z)] = 0. \quad (3.67)$$

All these relations will be useful in the explicit construction of the space  $W$  from the operators  $\mathcal{Z}(\epsilon|n)$ . This is illustrated in the next section.

## 4 Explicit constructions of some $U_q(\widehat{sl(2)})$ -modules

Let us briefly recall the definition of some  $U_q(\widehat{sl(2)})$ -modules [18, 2, 11]. For this, we still need some notions from  $\widehat{sl(2)}$  affine algebra, which is generated by  $\{e_i, f_i, h_i, d; i = 0, 1\}$ . We define on its Cartan subalgebra  $\hat{h} = \mathbb{C}h_0 + \mathbb{C}h_1 + \mathbb{C}d$  an invariant symmetric bilinear form  $(\ , \ )$  by

$$(h_i, h_j) = a_{i,j}, \quad (h_i, d) = \delta_{i,0}, \quad (d, d) = 0, \quad (4.68)$$

where

$$a_{i,i} = 2, \quad a_{i,1-i} = -2, \quad i = 0, 1, \quad (4.69)$$

are the elements of the  $\widehat{sl(2)}$  Cartan matrix. Let  $\hat{h}^* = \mathbb{C}\Lambda_0 + \mathbb{C}\Lambda_1 + \mathbb{C}\delta = \mathbb{C}\alpha_0 + \mathbb{C}\alpha_1 + \mathbb{C}\Lambda_0$  be the dual space to  $h$  with

$$\langle \Lambda_i, h_j \rangle = \delta_{i,j}, \quad \langle \delta, d \rangle = 1, \quad \langle \Lambda_i, d \rangle = 0, \quad \langle \delta, h_i \rangle = 0, \quad (4.70)$$

where

$$\langle \ , \ \rangle: \hat{h}^* \otimes \hat{h} \rightarrow \mathbb{C}. \quad (4.71)$$

is the natural pairing, the vectors  $\Lambda_i$  are the fundamental weights, the vectors  $\alpha_i$  are the positive roots and  $\delta = \alpha_0 + \alpha_1$  is the null root. One can induce a symmetric bilinear form  $(\ , \ )$  on  $\hat{h}^*$  by

$$(f(x), f(y)) = (x, y), \quad f(x), f(y) \in \hat{h}^* \quad x, y \in \hat{h}. \quad (4.72)$$

Here for a given  $x \in \hat{h}$ ,  $f(x)$  is defined to be the unique vector in  $\hat{h}^*$  such that

$$\langle f(x), y \rangle = (x, y), \quad \forall y \in \hat{h}. \quad (4.73)$$

This relation allows the identification of any element in  $\hat{h}$  with a unique element in  $\hat{h}^*$ . In particular, the elements  $h_i$  and  $d$  of the Cartan subalgebra are identified with  $\alpha_i$  and  $\Lambda_0$  respectively. Note that this identification should not be confused with the identification  $h_1 = \alpha(0)$ , which is the classical analogue of the identification that we have made implicitly in  $K = q^{h_1} = q^{\alpha_1(0)} = q^{\alpha(0)}$  in the case of  $U_q(\widehat{sl(2)})$  (see (2.14)). To see the relation between the two identifications, let  $v_\lambda$  be a weight vector of  $\hat{h}$  with weight  $\lambda$  then

$$h_1 v_\lambda = \langle \lambda, h_1 \rangle v_\lambda \equiv \langle \lambda, \alpha(0) \rangle v_\lambda \equiv (\lambda, \alpha_1) v_\lambda \equiv (\lambda, \alpha) v_\lambda. \quad (4.74)$$

More explicitly the symmetric bilinear form on  $\hat{h}^*$  is given by

$$\begin{aligned} (\Lambda_i, \Lambda_j) &= \frac{1}{2} \delta_{i,1} \delta_{j,1}, \quad (\Lambda_i, \delta) = 1, \quad (\delta, \delta) = 0, \\ (\alpha_i, \alpha_j) &= a_{i,j}, \quad (\alpha_i, \Lambda_0) = \delta_{i,0}, \quad (\Lambda_0, \Lambda_0) = 0, \quad i, j = 0, 1. \end{aligned} \quad (4.75)$$

The weights  $\lambda \in \hat{h}^*$  such that

$$\lambda = n_0 \Lambda_0 + n_1 \Lambda_1, \quad n_0, n_1 \in \mathbb{N}, \quad (4.76)$$

are called dominant integral weights, and  $n_0 + n_1 = k$  is the level that we have introduced previously.

As defined in Section 2, the algebra  $U_q(\widehat{sl(2)})$  is generated by  $\{e_i, f_i, K^{\pm 1}, \gamma^{\pm 1}, q^{\pm d}; i = 0, 1\}$ . Let  $V$  be a  $U_q(\widehat{sl(2)})$ -module and  $\mu \in \hat{h}$ , the subspace  $V_\mu \subset V$  defined by

$$V_\mu = \{v \in V / K^{\pm 1} v = q^{\pm \langle \mu, h_1 \rangle} v, \quad \gamma^{\pm 1} v = q^{\pm k} v, \quad q^{\pm d} v = q^{\pm \langle \mu, d \rangle} v\}, \quad (4.77)$$

is called a  $\mu$ -weight space, and any  $v \in V_\mu$  is referred to as a  $\mu$ -weight vector. The module  $V$  becomes a weight module if it is the direct sum of its weight spaces. A  $U_q(\widehat{sl(2)})$ -highest weight vector  $v_\lambda$  in  $V$  is a  $\lambda$ -weight vector which satisfies the additional condition

$$e_i v_\lambda = 0, \quad i = 0, 1. \quad (4.78)$$

The space  $V$  is called a  $U_q(\widehat{sl(2)})$ -highest weight module if it generated from a  $\lambda$ -highest weight vector  $v_\lambda$ . In this case,  $V$  is also a  $U_q(\widehat{sl(2)})$  weight module and  $v_\lambda$  is unique (up to a multiplication by a scalar), and hence we label  $V$  by the weight  $\lambda$  as  $V(\lambda)$ . The  $U_q(\widehat{sl(2)})$ -module  $V(\lambda)$  is called standard if it is generated from a highest weight vector  $v_\lambda$  with a dominant integral weight  $\lambda$  and such that

$$f_i^{<\lambda, h_i>+1} v_\lambda = 0, \quad i = 0, 1; \quad (4.79)$$

in which case it is irreducible, and thus called also an irreducible highest weight module. Let  $U_q(sl(2))$  be a subalgebra of  $U_q(\widehat{sl(2)})$  generated by  $\{e_1, f_1, K^{\pm 1}\}$  and let  $M = \mathbb{C}v_0$  be the trivial one-dimensional  $U_q(sl(2))$ -weight module. We also introduce  $U_q(\widehat{sl(2)})_{\geq 0}$ ,  $U_q(\widehat{sl(2)})_{>0}$  and  $U_q(\widehat{sl(2)})_{<0}$  as three subalgebras of  $U_q(\widehat{sl(2)})$  generated by all elements with nonnegative, positive and negative degrees with respect to  $q^d$ , respectively. We equip  $M$  with a  $U_q(\widehat{sl(2)})_{\geq 0}$ -module structure by

$$\begin{aligned} q^{\pm d} v_0 &= q^{\pm a} v_0, \quad a \in \mathbb{C}, \\ \gamma^{\pm 1} v_0 &= q^{\pm k} v_0, \\ xv_0 &= 0, \quad \forall x \in U_q(\widehat{sl(2)})_{>0}. \end{aligned} \quad (4.80)$$

Here  $a$  is scalar, which can be set to 0, without loss of generality. From  $M$ , we induce the following  $U_q(\widehat{sl(2)})$ -module:

$$G(M) = U_q(\widehat{sl(2)}) \otimes_{U_q(\widehat{sl(2)})_{\geq 0}} M, \quad (4.81)$$

which is called a generalized Verma module [10, 11]. In fact, since  $M$  is a one-dimensional  $U_q(sl(2))$ -module,  $G(M)$  is the simplest example of a  $U_q(\widehat{sl(2)})$ -generalized Verma module. Slightly more complicated example of generalized Verma modules can be constructed from higher dimensional  $M$  modules. We define the module  $W(M)$  through the following isomorphism of  $U_q(\widehat{sl(2)})$ -modules:

$$G(M) \simeq U_q(\widehat{sl(2)})_{<0} \otimes_{\mathbb{C}} M \simeq S(\hat{h}^-) \otimes_{\mathbb{C}} W(M). \quad (4.82)$$

Both modules  $G(M)$  and  $W(M)$  are  $q^d$ -weight modules with weight space decompositions:

$$\begin{aligned} G(M) &= \oplus_{n \leq 0} G(M)_n, \\ W(M) &= \oplus_{n \leq 0} W(M)_n. \end{aligned} \quad (4.83)$$

From the work of Lusztig [19], we know that for generic  $q$  the dimensions of weight spaces of  $U_q(\widehat{sl(2)})$ -modules are the same as those of the weight spaces of the  $\widehat{sl(2)}$ -modules. The characters  $\chi(G(M))$  and  $\chi(W(M))$  of  $G(M)$  and  $W(M)$  have been computed in the  $\widehat{sl(2)}$  case in [11, 6] and are given by:

$$\begin{aligned}\chi(G(M)) &= \sum_{n \geq 0} \dim(G(M)_n) p^n = \frac{1}{\prod_{n > 0} (1 - p^n)^3}, \\ \chi(W(M)) &= \sum_{n \geq 0} \dim(W(M)_n) p^n = \frac{1}{\prod_{n > 0} (1 - p^n)^2},\end{aligned}\tag{4.84}$$

where  $p \in \mathbb{C}^*$  is a formal variable. The second character will allow us to prove the linear independence of a particular set of vectors constructed from the  $\mathcal{Z}_q$  operators  $\mathcal{Z}(\epsilon|n)$  and which span  $W(M)$ . This means that the latter set of vectors is a basis for  $W(M)$ .

Let us now address the explicit constructions of the standard modules in the cases  $k = 1$  and  $k = 2$ . As explained previously, we should address only the explicit constructions of the space  $W$  in terms of the  $\mathcal{Z}_q$  operators since the  $S(\hat{h}^-)$  part is already constructed in terms of polynomials of  $\alpha(n)$ . Consequently, in the sequel we will mainly concentrate on the  $W$  part of the  $U_q(\widehat{sl(2)})$ -modules.

Case I:  $k = 1$

In this case, the relations (3.61)-(3.67) satisfied by the  $\mathcal{Z}_q$  operators  $\mathcal{Z}(\epsilon|z)$  simplify significantly and reduce to the following relations:

$$\begin{aligned}& \frac{\mathcal{Z}(\epsilon|z)\mathcal{Z}(-\epsilon|w)}{(1 - q^{-1}wz^{-1})(1 - qwz^{-1})} - \frac{\mathcal{Z}(-\epsilon|w)\mathcal{Z}(\epsilon|z)}{(1 - q^{-1}zw^{-1})(1 - qzw^{-1})} \\ &= \frac{\epsilon}{q - q^{-1}} \left( \Psi_0 \delta(zw^{-1}q^{-\epsilon}) - \Phi_0 \delta(zw^{-1}q^{\epsilon}) \right) \\ &= \frac{1}{q - q^{-1}} \left( q^{\epsilon\alpha(0)} \delta(zw^{-1}q^{-1}) - q^{-\epsilon\alpha(0)} \delta(zw^{-1}q) \right),\end{aligned}\tag{4.85}$$

$$w^2 \mathcal{Z}(\epsilon|z) \mathcal{Z}(\epsilon|w) = z^2 \mathcal{Z}(\epsilon|w) \mathcal{Z}(\epsilon|z),\tag{4.86}$$

$$[\alpha(n), \mathcal{Z}(\epsilon|z)] = 0, \quad n \in \mathbb{Z} \setminus \{0\},\tag{4.87}$$

$$\Psi(0) \mathcal{Z}(\epsilon|z) = q^{2\epsilon} \mathcal{Z}(\epsilon|z) \Psi(0),\tag{4.88}$$

$$\Phi(0) \mathcal{Z}(\epsilon|z) = q^{-2\epsilon} \mathcal{Z}(\epsilon|z) \Phi(0),\tag{4.89}$$

$$X(\epsilon|z) = S_{-\epsilon}^-(zq^{-\epsilon k}) S_{-\epsilon}^+(zq^{\epsilon k}) \otimes \mathcal{Z}(\epsilon|z),\tag{4.90}$$

$$q^d \mathcal{Z}(\epsilon|z) = \mathcal{Z}(\epsilon|zq^{-1}) q^d,\tag{4.91}$$



$$[\gamma^\pm, \mathcal{Z}(\epsilon|z)] = 0. \quad (4.92)$$

We now construct the appropriate space  $W$  that is compatible with the above relations. We start with (4.88) and (4.89). These relations imply that  $\mathcal{Z}(\epsilon|z)$  cannot be trivial, i.e., it does not act like a scalar which can be rescaled to 1 on  $W$ , and hence  $W$  cannot be a trivial one-dimensional space. If the latter statement were not true, the relations (3.50) would mean that  $X(\epsilon|z)$  acts trivially also on  $S(\hat{h}^-)$  because of the fact that both  $\Psi_0$  and  $\Phi_0$  do act like the identity on  $S(\hat{h}^-)$ . But if  $X(\epsilon|z)$  acts trivially on  $S(\hat{h}^-)$ , the relation (2.23) implies that  $\Psi(z)$  and  $\Phi(z)$  also act trivially on  $S(\hat{h}^-)$ , which is impossible from the definition itself of  $S(\hat{h}^-)$ . Therefore  $\mathcal{Z}(\epsilon|z)$  cannot act like the identity on  $W$ . To make it act nontrivially and consistently with  $\Psi_0$  and  $\Phi_0$ , let us write the positive simple root  $\alpha_1 = \alpha$  and introduce the  $sl(2)$  root lattice  $Q = \mathbb{Z}\alpha$  and the weight lattice  $P = \mathbb{Z}\alpha/2$ , which decomposes as  $P = Q \cup (Q + \alpha/2)$ . Denote by  $\mathbb{C}[P]$  the group algebra with the basis  $\{e^\beta, \beta \in P\}$  and the commutative multiplication

$$e^\beta e^\gamma = e^\gamma e^\beta = e^{\beta+\gamma}, \quad \beta, \gamma \in [P]. \quad (4.93)$$

Obviously, we have  $\mathbb{C}[Q] \subset \mathbb{C}[P]$ . We define the action of  $\alpha(0) \in \text{End}(\mathbb{C}[P])$  on  $\mathbb{C}[P]$  by

$$\alpha(0) : \quad e^\beta \rightarrow (\alpha, \beta) e^\beta. \quad (4.94)$$

This implies that the formal Laurent series  $z^{\alpha(0)} \in \text{End}(\mathbb{C}[Q] \otimes [z, z^{-1}])$  acts on  $\mathbb{C}[P]$  as

$$z^{\alpha(0)} : \quad e^\beta \rightarrow z^{(\alpha, \beta)} e^\beta. \quad (4.95)$$

It can easily be checked that the latter two relations are equivalent to

$$\begin{aligned} [\alpha(0), e^\beta] &= (\alpha, \beta) e^\beta, \\ z^{\alpha(0)} e^\beta &= z^{(\alpha, \beta)} e^\beta z^{\alpha(0)}, \end{aligned} \quad (4.96)$$

respectively. We define also the action of  $q^d$  on  $\mathbb{C}[P]$  (i.e.,  $\mathbb{C}[P]$  is a graded space) by

$$q^d : \quad e^\beta \rightarrow e^{-1/2(\beta, \beta)} e^\beta, \quad \beta \in \mathbb{C}[P], \quad (4.97)$$

that is,

$$q^d e^\beta = e^\beta q^d q^{-\beta(0) - (\beta, \beta)/2}. \quad (4.98)$$

From the relations (4.88), (4.89) and (4.96) we see that as a candidate for  $\mathcal{Z}(\epsilon|z)$  we can take

$$\mathcal{Z}(\epsilon|z) = e^{\epsilon\alpha}, \quad \epsilon = \pm, \quad (4.99)$$

where  $\alpha$  is the  $sl(2)$  positive simple root. However, according to the relation (4.91) this candidate does not have the right degree since it does not depend on the formal variable  $z$ . This is corrected by taking instead the following candidate for  $\mathcal{Z}(\epsilon|z)$ :

$$\mathcal{Z}(\epsilon|z) = e^{\epsilon\alpha} z^{\epsilon\alpha(0) + \frac{1}{2}(\alpha, \alpha)} = e^{\epsilon\alpha} z^{\epsilon\alpha(0) + 1}, \quad \epsilon = \pm. \quad (4.100)$$

With this choice, it can easily be verified that all the remaining relations are satisfied. The less trivial one is (4.85), which can be shown as follows:

$$\begin{aligned} & \mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|w) \left( \frac{1}{(1 - q^{-1}wz^{-1})(1 - qwz^{-1})} - \frac{z^2 w^{-2}}{(1 - q^{-1}zw^{-1})(1 - qzw^{-1})} \right) \\ &= \mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|w) \left( \frac{1}{1 - qwz^{-1}} \delta(zw^{-1}q^{-1}) - \frac{qzw^{-1}}{1 - qzw^{-1}} \delta(zw^{-1}q) \right) \\ &= \frac{q\mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|zq^{-1}) \delta(zw^{-1}q^{-1}) - q^{-1} \mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|zq) \delta(zw^{-1}q)}{q - q^{-1}} \\ &= \frac{1}{q - q^{-1}} \left( q^{\epsilon\alpha(0)} \delta(zw^{-1}q^{-1}) - q^{-\epsilon\alpha(0)} \delta(zw^{-1}q) \right), \end{aligned} \quad (4.101)$$

where we have used (4.96), (2.31), and

$$\mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|w) = (zw^{-1})^{\epsilon\alpha(0)-1}. \quad (4.102)$$

As a conclusion,  $X(\epsilon|z)$  acts on  $S(\hat{h}^-) \otimes \mathbb{C}[P]$  as  $S_{-\epsilon}^-(zq^{-\epsilon}) S_{-\epsilon}^+(zq^{\epsilon}) \otimes e^{\epsilon\alpha} z^{\epsilon\alpha(0)+1}$  and is single-valued. Moreover, the subspaces  $S(\hat{h}^-) \otimes \mathbb{C}[Q]$  and  $S(\hat{h}^-) \otimes e^{\frac{\alpha}{2}} \mathbb{C}[Q]$  whose direct sum is  $S(\hat{h}^-) \otimes \mathbb{C}[P]$  are invariant and irreducible. They are in fact isomorphic to the standard (basic) modules  $V(\Lambda_0)$  and  $V(\Lambda_1)$  with highest weight vectors realized as  $1 \otimes 1$  and  $1 \otimes e^{\frac{\alpha}{2}}$ , respectively.

Case II:  $k = 2$

In this case, the relations (3.61)-(3.67) satisfied by the  $\mathcal{Z}_q$  operators reduce to

$$\begin{aligned} & \frac{\mathcal{Z}(\epsilon|z) \mathcal{Z}(-\epsilon|w)}{1 - wz^{-1}} - \frac{\mathcal{Z}(-\epsilon|w) \mathcal{Z}(\epsilon|z)}{1 - zw^{-1}} = \frac{\epsilon}{q - q^{-1}} \left( \Psi_0 \delta(zw^{-1}q^{-2\epsilon}) - \Phi_0 \delta(zw^{-1}q^{2\epsilon}) \right) \\ &= \frac{1}{q - q^{-1}} \left( q^{\epsilon\alpha(0)} \delta(zw^{-1}q^{-2}) - q^{-\epsilon\alpha(0)} \delta(zw^{-1}q^2) \right), \end{aligned} \quad (4.103)$$

$$(z - wq^{2\epsilon})(1 - wz^{-1}q^{-2\epsilon})\mathcal{Z}(\epsilon|z)\mathcal{Z}(\epsilon|w) = (zq^{2\epsilon} - w)(1 - zw^{-1}q^{-2\epsilon})\mathcal{Z}(\epsilon|w)\mathcal{Z}(\epsilon|z), \quad (4.104)$$

$$[\alpha(n), \mathcal{Z}(\epsilon|z)] = 0, \quad n \in \mathbb{Z} \setminus \{0\}, \quad (4.105)$$

$$\Psi(0)\mathcal{Z}(\epsilon|z) = q^{2\epsilon}\mathcal{Z}(\epsilon|z)\Psi(0), \quad (4.106)$$

$$\Phi(0)\mathcal{Z}(\epsilon|z) = q^{-2\epsilon}\mathcal{Z}(\epsilon|z)\Phi(0), \quad (4.107)$$

$$X_\epsilon(z) = S_{-\epsilon}^-(zq^{-2\epsilon})S_{-\epsilon}^+(zq^{2\epsilon}) \otimes \mathcal{Z}(\epsilon|z), \quad (4.108)$$

$$q^d\mathcal{Z}(\epsilon|z) = \mathcal{Z}(\epsilon|zq^{-1})q^d, \quad (4.109)$$

$$[\gamma^\pm, \mathcal{Z}(\epsilon|z)] = 0. \quad (4.110)$$

As in the case  $k = 1$ , in order to satisfy the relations (4.106) and (4.107) we need to consider the group algebras  $\mathbb{C}[P]$  and  $\mathbb{C}[Q]$ . The operators  $\alpha(0)$  and  $z^{\alpha(0)}$  act on them in the same manner as in the case  $k = 1$  but now  $q^d$  acts on them slightly differently. Its action is rescaled by a factor  $2(=k)$  as:

$$q^d : e^\beta \rightarrow e^{-1/4(\beta,\beta)}e^\beta, \quad \beta \in \mathbb{C}[P], \quad (4.111)$$

which is equivalent to

$$q^d e^\beta = e^\beta q^d q^{-\frac{\beta(0)}{2} - (\beta,\beta)/4}, \quad \beta \in \mathbb{C}[P]. \quad (4.112)$$

The candidate for  $\mathcal{Z}(\epsilon|z)$  that is compatible with this action of  $q^d$ , and the relations (4.106) and (4.107) is given by

$$\mathcal{Z}(\epsilon|z) = e^{\epsilon\alpha} z^{\epsilon\frac{\alpha(0)}{2} + \frac{1}{4}(\alpha,\alpha)} = e^{\epsilon\alpha} z^{\epsilon\frac{\alpha(0)}{2} + \frac{1}{2}}, \quad \epsilon = \pm. \quad (4.113)$$

It is also trivial that this candidate satisfies both relations (4.105) and (4.110). However, it can easily be checked that the remaining relations (4.103) and (4.104) are not satisfied with this candidate. This means that the group algebra  $\mathbb{C}[P]$  is not big enough and therefore we must enlarge it with at least an extra algebra, which will turn out to be the Clifford algebra. To see this, let  $\psi(\epsilon|z)$  be a generating function of the elements of yet an unknown quantum

deformation of the Clifford algebra such that

$$\begin{aligned}
[\alpha(n), \psi(\epsilon|z)] &= 0, \\
[\Psi_0, \psi(\epsilon|z)] &= 0, \\
[\Phi_0, \psi(\epsilon|z)] &= 0, \\
q^d \psi(\epsilon|z) &= \psi(\epsilon|z q^{-1}) q^d, \\
\gamma \psi(\epsilon|z) &= \psi(\epsilon|z) \gamma.
\end{aligned} \tag{4.114}$$

These equations guarantee that the following candidate for  $\mathcal{Z}(\epsilon|z)$  still preserves all the relations that are already satisfied by our first candidate (4.113):

$$\mathcal{Z}(\epsilon|z) = e^{\epsilon\alpha} z^{\epsilon\frac{\alpha(0)}{2} + \frac{1}{4}(\alpha, \alpha)} \psi(\epsilon|z) = e^{\epsilon\alpha} z^{\epsilon\frac{\alpha(0)}{2} + \frac{1}{2}} \psi(\epsilon|z), \quad \epsilon = \pm. \tag{4.115}$$

In terms of these fields  $\psi(\epsilon|z)$ , the remaining two relations which must also be satisfied take the following form after using (4.96):

$$\begin{aligned}
&\frac{1}{q - q^{-1}} \left( q^{\epsilon\alpha(0)} \delta(zw^{-1}q^{-2}) - q^{-\epsilon\alpha(0)} \delta(zw^{-1}q^2) \right), \\
&= (zw^{-1})^{-\frac{1}{2}} (zw^{-1})^{\frac{\epsilon}{2}\alpha(0)} \left( \frac{\psi(\epsilon|z)\psi(-\epsilon|w)}{1 - wz^{-1}} - \frac{zw^{-1}\psi(-\epsilon|w)\psi(\epsilon|z)}{1 - zw^{-1}} \right) \\
&= (zw^{-1})^{-\frac{1}{2}} (zw^{-1})^{\frac{\epsilon}{2}\alpha(0)} \frac{\{\psi(\epsilon|z), \psi(-\epsilon|w)\}}{1 - wz^{-1}},
\end{aligned} \tag{4.116}$$

$$z(z - wq^{2\epsilon})(1 - wz^{-1}q^{-2\epsilon})\psi(\epsilon|z)\psi(\epsilon|w) = w(zq^{2\epsilon} - w)(1 - zw^{-1}q^{-2\epsilon})\psi(\epsilon|w)\psi(\epsilon|z), \tag{4.117}$$

where  $\{A, B\} = AB + BA$  is the anticommutator of  $A$  and  $B$ . Clearly, (4.116) is satisfied if the following conditions on  $\psi(\epsilon|z)$  hold:

$$\{\psi(\epsilon|z), \psi(-\epsilon|w)\} = \delta(zw^{-1}q^{-2}) + \delta(zw^{-1}q^2), \quad \text{if } z^{\frac{\alpha(0)}{2}} \in \text{End}(e^{\frac{\alpha}{2}}\mathbb{C}[Q] \otimes \mathbb{C}[z, z^{-1}]), \tag{4.118}$$

$$\{\psi(\epsilon|z), \psi(-\epsilon|w)\} = (zw^{-1})^{\frac{1}{2}}(q^{-1}\delta(zw^{-1}q^{-2}) + q\delta(zw^{-1}q^2)), \quad \text{if } z^{\frac{\alpha(0)}{2}} \in \text{End}(\mathbb{C}[Q]) \otimes \mathbb{C}[z, z^{-1}]. \tag{4.119}$$

Since the right hand sides of these equations do not depend on  $\epsilon$  we might try to identify  $\psi(+|z)$  and  $\psi(-|z)$ . Relation (4.118) means that  $\psi(z) \equiv \psi(\epsilon|z)$  has the Laurent expansion

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n}, \quad \epsilon = \pm. \tag{4.120}$$

Substituting this expansion back in (4.118) and comparing the coefficients of powers of  $zw^{-1}$ , we obtain the following anticommutation relations for the modes  $\psi_n$ :

$$\{\psi_n, \psi_m\} = (q^{2n} + q^{-2n})\delta_{n+m,0}, \quad \text{if } z^{\frac{\alpha(0)}{2}} \in \text{End}(e^{\frac{\alpha}{2}}\mathbb{C}[Q] \otimes \mathbb{C}[z, z^{-1}]). \quad (4.121)$$

This is the usual Rammond (R) sector for the modes  $\psi_n; n \in \mathbb{Z}$ . Similarly (4.119) enforces the following expansion:

$$\psi(z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \psi_r z^{-r}, \quad \epsilon = \pm. \quad (4.122)$$

As in the (R) sector, relation (4.119) leads to the following anticommutation relations for the modes  $\psi_r$ :

$$\{\psi_r, \psi_s\} = (q^{2r} + q^{-2r})\delta_{r+s,0}, \quad \text{if } z^{\frac{\alpha(0)}{2}} \in \text{End}(\mathbb{C}[Q] \otimes \mathbb{C}[z, z^{-1}]). \quad (4.123)$$

This is the familiar Neveu-Schwarz (NS) sector for the modes  $\psi_r$ . Note that both relations (4.118) and (4.119) imply that

$$\psi(z)\psi(z) = \delta(q^2), \quad (4.124)$$

which means that unlike in the classical case, the divergence in the product  $\psi(z)\psi(z)$  at the same “point”  $z$  is regularized by the deformation parameter, which therefore can be thought of as a regularization parameter as well. However, this does not mean that the product  $\psi(z)\psi(w)$  is divergence-free at arbitrary points  $z$  and  $w$ . In fact, according to (4.118) and (4.119), it is divergent at  $z = wq^{\pm 2}$ , and therefore we still need to regularize it by extra means, other than the parameter  $q$ , to make its action well defined on its modules. This regularization can be achieved through the normal ordering. As usual, a product of operators  $\psi_n$  ( $\psi_r$ ) with  $n \in \mathbb{Z}$  ( $r \in \mathbb{Z} + 1/2$ ) is normal ordered if all  $\psi_n$  ( $\psi_r$ ) with  $n < 0$  ( $r \leq -1/2$ ) are moved to the left of all  $\psi_n$  ( $\psi_r$ ) with  $n > 0$  ( $r \geq 1/2$ ), and in the (R) sector  $\psi_0^2$  is normal ordered in such a way that the normal ordered product  $\psi(z)\psi(z)$  is null, that is,  $\psi_0^2 = 0$ . Note that the same reasoning about the notion of normal ordering must also be applied to products of the operators  $\alpha(n)$  and  $a$  with  $n \in \mathbb{Z}$  and  $a \in \mathbb{C}[P]$  to make the action of products of  $X(\epsilon|z)$  free of divergence. In this case the normal ordering consists in moving all  $\alpha(n)$  with  $n < 0$  to the left of all  $\alpha(n)$  with  $n > 0$ , and  $a$  to the left of  $\alpha(0)$ . Therefore, products of  $\alpha(n)$  with  $n \in \mathbb{Z}^*$  in  $X(\epsilon|z)$  as defined by (3.55) are already normal ordered, and

hence the action of  $X(\epsilon|z)$  on the symmetric algebra  $S(\hat{h}^-)$  is well defined by construction. The remaining relation (4.117) is equivalent to

$$\begin{aligned}\{\psi_{n+2}, \psi_m\} &= (q^2 + q^{-2})\{\psi_{n+1}, \psi_{m+1}\}, \quad n, m \in \mathbb{Z}, \\ \{\psi_{r+2}, \psi_s\} &= (q^2 + q^{-2})\{\psi_{r+1}, \psi_{s+1}\}, \quad r, s \in \mathbb{Z} + \frac{1}{2},\end{aligned}\tag{4.125}$$

and can easily be verified using (4.121) and (4.123). The Clifford algebra with the anticommutation relations (4.121) and (4.123) first appeared in the work of Bernard on the explicit construction of  $U_q(\mathfrak{so}(2n+1))$ -standard modules with level  $k=1$  [9]. Then, it was used in [18, 20] for the calculation of the  $N$ -point correlation functions for the spin-1 XXZ model.

Let  $T^R(T^{NS})$ ,  $T_{even}^R(T_{even}^{NS})$ ,  $T_{odd}^R(T_{odd}^{NS})$  be the Fock space spanned by the modes  $\{\psi_n, n < 0\}$  ( $\{\psi_r, r < 0\}$ ), a subspace of  $T^R(T^{NS})$  spanned by an even number of modes  $\psi_n$  ( $\psi_r$ ), and a subspace of  $T^R(T^{NS})$  spanned by an odd number of modes  $\psi_n$  ( $\psi_r$ ) respectively. Note that in the (R) sector the zero mode  $\psi_0$  acts trivially on  $T^R$ , and so to make its action non-trivial we extend the Fock space  $T^R$  by  $\mathbb{C}^2$ , with basis  $\{v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$  such that  $\psi_n(n \neq 0)$  and  $\psi_0$  act as  $\psi_n \otimes 1$  and  $1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  on  $T^R \otimes \mathbb{C}^2$  respectively. Putting all the pieces together, we conclude that  $X(\epsilon|z)$  acts as

$$X(\epsilon|z) = S_{-\epsilon}^-(zq^{-2\epsilon})S_{-\epsilon}^+(zq^{2\epsilon}) \otimes \psi(z) \otimes e^{\epsilon\alpha} z^{\epsilon\frac{\alpha(0)}{2} + \frac{1}{2}}\tag{4.126}$$

on the space  $S(\hat{h}^-) \otimes (T^R \otimes \mathbb{C}^2) \otimes e^{\frac{\alpha}{2}}\mathbb{C}[Q]$  in the (R) sector, and on the space  $S(\hat{h}^-) \otimes (T^{NS}) \otimes \mathbb{C}[Q]$  in the (NS) sector. Moreover, the  $U_q(\widehat{\mathfrak{sl}(2)})$ -standard modules  $V(2\Lambda_0)$ ,  $V(2\Lambda_1)$  and  $V(\Lambda_0 + \Lambda_1)$  are isomorphic to the following subspaces of the latter spaces

$$\begin{aligned}V(2\Lambda_0) &\sim S(\hat{h}^-) \otimes T_{even}^{NS} \otimes \mathbb{C}[2Q] \oplus S(\hat{h}^-) \otimes T_{odd}^{NS} \otimes e^\alpha \mathbb{C}[2Q], \\ V(2\Lambda_1) &\sim S(\hat{h}^-) \otimes T_{even}^{NS} \otimes e^\alpha \mathbb{C}[2Q] \oplus S(\hat{h}^-) \otimes T_{odd}^{NS} \otimes \mathbb{C}[2Q], \\ V(\Lambda_0 + \Lambda_1) &\sim S(\hat{h}^-) \otimes (T_{even}^R \otimes v_+ \oplus T_{odd}^R \otimes v_-) \otimes e^{\frac{\alpha}{2}}\mathbb{C}[2Q] \\ &\quad \oplus S(\hat{h}^-) \otimes (T_{odd}^R \otimes v_+ \oplus T_{even}^R \otimes v_-) \otimes e^{\frac{3\alpha}{2}}\mathbb{C}[2Q],\end{aligned}\tag{4.127}$$

and their respective highest weight vectors are given by:

$$\begin{aligned}v_{2\Lambda_0} &= 1 \otimes 1 \otimes 1, \\ v_{2\Lambda_1} &= 1 \otimes 1 \otimes e^\alpha, \\ v_{\Lambda_0 + \Lambda_1} &= 1 \otimes 1 \otimes v_+ \otimes e^{\frac{\alpha}{2}}.\end{aligned}\tag{4.128}$$

Case III:  $G(M)$  with arbitrary nonzero  $k$ :

In this case, we will not split the  $\mathcal{Z}_q$  algebra into a tensor product of a group algebra and a new algebra, which is parafermionic in nature and has been partially described in [17]. We will rather construct the  $W(M)$  module in terms of the  $\mathcal{Z}_q$  operators themselves. This will be sufficient to find an explicit realization of the  $U_q(\widehat{sl(2)})$ -generalized Verma module  $G(M) = S(\hat{h}^-) \otimes W(M)$  since the  $S(\hat{h}^-)$  is already constructed in terms of symmetric polynomials of  $\alpha(n)$ . To this end, we first define

$$\mathcal{Z}(\epsilon, \epsilon'|z, w) = f(\epsilon, \epsilon'|wz^{-1})\mathcal{Z}(\epsilon|z)\mathcal{Z}(\epsilon'|w), \quad (4.129)$$

$$f(\epsilon, \epsilon'|z) = \frac{(q^{-(\epsilon+\epsilon')k/2+k-2}z; q^{2k})_{\infty}^{\epsilon\epsilon'}}{(q^{-(\epsilon+\epsilon')k/2+k+2}z; q^{2k})_{\infty}^{\epsilon\epsilon'}}. \quad (4.130)$$

Next, following some ideas in [6] in the case of  $\widehat{sl(2)}$  algebra, and which have been well summarized in [21] in the case of the elliptic algebra, we introduce the following formal Laurent and power series:

$$\mathcal{Z}(\epsilon, \epsilon'|z, w) = \sum_{n_1, n_2 \in \mathbb{Z}} \mathcal{Z}(\epsilon, \epsilon'|n_1, n_2) z^{-n_1} w^{-n_2}, \quad (4.131)$$

$$\mathcal{Z}(\epsilon|z) = \sum_{n \in \mathbb{Z}} \mathcal{Z}(\epsilon|n) z^{-n}, \quad (4.132)$$

$$f(\epsilon, \epsilon'|z) = \frac{1}{\sum_{n \geq 0} \tilde{a}_n^{\epsilon, \epsilon'} z^{-n}} = \sum_{n \geq 0} a_n^{\epsilon, \epsilon'} z^{-n}. \quad (4.133)$$

Relation (4.130) and (4.133) imply that

$$\begin{aligned} a_0^{\epsilon, \epsilon'} &= \tilde{a}_0^{\epsilon, \epsilon'} = 1, \\ \sum_{n \geq 0} \tilde{a}_n^{\epsilon, \epsilon'} a_{m-n}^{\epsilon, \epsilon'} &= \delta_{m,0}, \quad m \geq 0, \\ a_n^{\epsilon, \epsilon'} &= \tilde{a}_n^{\epsilon, \epsilon'} = 0, \quad n < 0. \end{aligned} \quad (4.134)$$

Substituting the above expansions back in (4.129), and comparing the coefficients of the powers of  $wz^{-1}$ , we obtain:

$$\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(\epsilon'|n_2) = \sum_{n \geq 0} \tilde{a}_n^{\epsilon, \epsilon'} \mathcal{Z}(\epsilon, \epsilon'|n_1 - n, n_2 + n), \quad (4.135)$$

$$\mathcal{Z}(\epsilon, \epsilon'|n_1, n_2) = \sum_{n \geq 0} a_n^{\epsilon, \epsilon'} \mathcal{Z}(\epsilon|n_1 - n) \mathcal{Z}(\epsilon'|n_2 + n). \quad (4.136)$$

Furthermore, substituting the latter expansions in the quantum generalized relations (3.61) and (3.61) and using (4.136), we arrive at

$$\mathcal{Z}(\epsilon, -\epsilon|n_1, n_2) = \mathcal{Z}(-\epsilon, \epsilon|n_2, n_1) + Y(\epsilon|n_1)\delta_{n_1+n_2,0}, \quad (4.137)$$

$$\mathcal{Z}(\epsilon, \epsilon|n_1, n_2) = q^{2\epsilon}\mathcal{Z}(\epsilon, \epsilon|n_1 - 1, n_2 + 1) + q^{2\epsilon}\mathcal{Z}(\epsilon, \epsilon|n_2, n_1) - \mathcal{Z}(\epsilon, \epsilon|n_2 + 1, n_1 - 1), \quad (4.138)$$

respectively, and where

$$Y(\epsilon|n) = \frac{1}{q - q^{-1}}(q^{kn+\epsilon\alpha(0)} - q^{-kn-\epsilon\alpha(0)}). \quad (4.139)$$

Relations (4.137) and (4.138) are useful in the normal ordering of products of  $\mathcal{Z}_q$  operators by moving any operator  $\mathcal{Z}(\epsilon|n_1)$  with  $n_1 > n_2$  to the right of  $\mathcal{Z}(\epsilon'|n_2)$ , and the operator  $\mathcal{Z}(+|n)$  to the right of  $\mathcal{Z}(-|n)$ . To see this let us examine the normal ordering of  $\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(\epsilon'|n_2)$  in the following three nontrivial cases:

A.  $n_1 > n_2$ ,  $\epsilon = -\epsilon'$ :

Using relations (4.135) and (4.137), we obtain

$$\begin{aligned} \mathcal{Z}(\epsilon|n_1)\mathcal{Z}(-\epsilon|n_2) &= \sum_{n \geq 0} \tilde{a}_n^{\epsilon, -\epsilon} \mathcal{Z}(\epsilon, -\epsilon|n_1 - n, n_2 + n) = \sum_{0 \leq n \leq \frac{n_1 - n_2 - x}{2}} \tilde{a}_n^{\epsilon, -\epsilon} \mathcal{Z}(-\epsilon, \epsilon|n_2 + n, n_1 - n) \\ &+ \sum_{0 \leq \frac{n_1 - n_2 - x}{2} < n} \tilde{a}_n^{\epsilon, -\epsilon} \mathcal{Z}(\epsilon, -\epsilon|n_1 - n, n_2 + n) + \sum_{0 \leq n \leq \frac{n_1 - n_2 - x}{2}} \tilde{a}_n^{\epsilon, -\epsilon} Y(\epsilon|n_1)\delta_{n_1+n_2,0}, \end{aligned} \quad (4.140)$$

where  $x$  is equal to 0 or 1 depending on whether  $n_1 - n_2$  is even or odd respectively. It is therefore clear from the latter relation and (4.136) that the product  $\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(-\epsilon|n_2)$  with  $n_1 > n_2$  can be normal ordered.

B.  $n_1 = n_2$ ,  $\epsilon = -\epsilon' = +$ :

Like the previous case, we have

$$\begin{aligned} \mathcal{Z}(+|n_1)\mathcal{Z}(-|n_1) &= \sum_{n \geq 0} \tilde{a}_n^{+, -} \mathcal{Z}(+, -|n_1 - n, n_1 + n), \\ &= \mathcal{Z}(-, +|n_1, n_1) + Y(+|0)\delta_{n_1,0} + \sum_{n > 0} \tilde{a}_n^{+, -} \mathcal{Z}(-\epsilon, \epsilon|n_1 - n, n_1 + n). \end{aligned} \quad (4.141)$$

The same argument as in case A holds, and hence the product  $\mathcal{Z}(+|n_1)\mathcal{Z}(-|n_1)$  can be normal ordered as well.



C.  $n_1 > n_2$ ,  $\epsilon = \epsilon'$ :

This case is less straightforward. First, relation (4.135) implies that

$$\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(\epsilon|n_2) = \sum_{n \geq 0} \tilde{a}_n^{\epsilon, \epsilon} \mathcal{Z}(\epsilon, \epsilon|n_1 - n, n_2 + n), \quad (4.142)$$

which according to (4.136) means that the product  $\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(\epsilon|n_2)$  can be normal ordered if we can normal order also any operator  $\mathcal{Z}(\epsilon, \epsilon|n_1, n_2)$  with  $n_1 > n_2$  by writing it as a linear combination of operators  $\mathcal{Z}(\epsilon, \epsilon|m_1, m_2)$  with  $m_1 \leq m_2$ . Relation (4.138) allows indeed this second type of normal ordering. The reason is that repeated use of this relation leads to

$$\begin{aligned} \mathcal{Z}(\epsilon, \epsilon|n_1 + 2p, n_1) &= q^{2\epsilon} \mathcal{Z}(\epsilon, \epsilon|n_1, n_1 + 2p) + q^{2(p-1)\epsilon} (q^{2\epsilon} - 1) \mathcal{Z}(\epsilon, \epsilon|n_1 + p, n_1 + p) \\ &\quad + \sum_{n=1}^{p-1} q^{2(n-1)\epsilon} (q^{4\epsilon} - 1) \mathcal{Z}(\epsilon, \epsilon|n_1 + n, n_1 + 2p - n), \quad p > 0, \\ \mathcal{Z}(\epsilon, \epsilon|n_1 + 2p + 1, n_1) &= q^{2\epsilon} \mathcal{Z}(\epsilon, \epsilon|n_1, n_1 + 2p + 1) \\ &\quad + \sum_{n=1}^p q^{2(n-1)\epsilon} (q^{4\epsilon} - 1) \mathcal{Z}(\epsilon, \epsilon|n_1 + n, n_1 + 2p + 1 - n), \quad p > 0, \end{aligned} \quad (4.143)$$

where all the operators  $\mathcal{Z}(\epsilon, \epsilon|m_1, m_2)$  in the right hand sides of the above equations have  $m_1 \leq m_2$ . Consequently, the product  $\mathcal{Z}(\epsilon|n_1)\mathcal{Z}(\epsilon|n_2)$  with  $n_1 > n_2$  can also be normal ordered.

Since  $X(\epsilon|z)$  acts as  $S_{-\epsilon}^-(zq^{-2\epsilon})S_{-\epsilon}^+(zq^{2\epsilon}) \otimes \mathcal{Z}(\epsilon|z)$  on  $S(\hat{h}^-) \otimes W(M) \otimes \mathbb{C}[z, z^{-1}]$  it is clear that  $W(M)$  is spanned by the vectors in the set

$$\{\mathcal{Z}(\epsilon_1|n_1) \dots \mathcal{Z}(\epsilon_s|n_s)v_0, \quad \epsilon_i = \pm, \quad n_i < 0, \quad i = 1, \dots, s; \quad s > 0\}. \quad (4.144)$$

The condition  $n_i < 0$  guarantees that the above vectors have negative degrees as they should (otherwise they are null) since  $S(\hat{h}^-) \otimes W(M)$  is a graded  $U_q(\widehat{sl(2)})$ -highest weight module. Because of the normal ordering of the  $\mathcal{Z}(\epsilon|n)$  operators discussed above, the above spanning set for  $W(G)$  can be reduced further to the smaller set

$$\begin{aligned} H &= \{\mathcal{Z}(\epsilon_1|n_1) \dots \mathcal{Z}(\epsilon_s|n_s)v_0, \quad \epsilon_i = \pm, \quad n_i \leq n_{i+1}; \\ &\quad \epsilon_i \leq \epsilon_{i+1} \quad \text{if} \quad n_i = n_{i+1}, \quad i = 1, \dots, s; \quad s > 0\}, \end{aligned} \quad (4.145)$$

where the order  $- < +$  is meant. It can easily be seen that the set  $H$  is a basis for  $W(M)$  since its character coincides with the one of  $W(G)$  as given by (4.84). Therefore, we have an explicit construction of the  $U_q(\widehat{sl(2)})$ -generalized Verma module  $G(M)$  with nonzero level  $k$ .

## 5 Relations in the $\mathcal{Z}_q$ enveloping algebra

In this section, we extend the quantum generalized commutation relations (3.61) and (3.61) to relations satisfied by arbitrary polynomials of  $Z(\epsilon|z)$ . For this purpose let us consider the following operators:

$$\mathcal{Z}(\epsilon_1, \dots, \epsilon_s | z_1, \dots, z_s) = S_{\epsilon_1}^-(z_1) \dots S_{\epsilon_s}^-(z_s) x_{\epsilon_1}(z_1) \dots x_{\epsilon_s}(z_s) S_{\epsilon_1}^+(z_1) \dots S_{\epsilon_s}^+(z_s), \quad s > 0, \quad (5.146)$$

which are a generalization of (3.47). They are expressed in terms of the operators  $\mathcal{Z}(\epsilon|z)$  introduced in (3.47) as:

$$Z(\epsilon_1, \dots, \epsilon_s | z_1, \dots, z_s) = \prod_{1 \leq i < j \leq s} \frac{(q^{-(\epsilon_i + \epsilon_j)k/2 + k - 2} z_j z_i^{-1}; q^{2k})_{\infty}^{\epsilon_i \epsilon_j}}{(q^{-(\epsilon_i + \epsilon_j)k/2 + k + 2} z_j z_i^{-1}; q^{2k})_{\infty}^{\epsilon_i \epsilon_j}} \mathcal{Z}(\epsilon_1 | z_1) \mathcal{Z}(\epsilon_2 | z_2) \dots \mathcal{Z}(\epsilon_s | z_s), \quad (5.147)$$

$$= \prod_{2 \leq i \leq s} \frac{(q^{-(\epsilon_1 + \epsilon_i)k/2 + k - 2} z_i z_1^{-1}; q^{2k})_{\infty}^{\epsilon_1 \epsilon_i}}{(q^{-(\epsilon_1 + \epsilon_i)k/2 + k + 2} z_i z_1^{-1}; q^{2k})_{\infty}^{\epsilon_1 \epsilon_i}} \mathcal{Z}(\epsilon_1 | z_1) \mathcal{Z}(\epsilon_2, \dots, \epsilon_s | z_2, \dots, z_s). \quad (5.148)$$

The above relations can easily be derived from

$$S_{\epsilon}^+(z_1) X_{\epsilon'}(z_2) = \frac{(q^{-(\epsilon + \epsilon')k/2 + k + 2} z_2 z_1^{-1}; q^{2k})_{\infty}^{\epsilon \epsilon'}}{(q^{-(\epsilon + \epsilon')k/2 + k - 2} z_2 z_1^{-1}; q^{2k})_{\infty}^{\epsilon \epsilon'}} X_{\epsilon'}(z_2) S_{\epsilon}^+(z_1), \quad (5.149)$$

$$S_{\epsilon}^-(z_1) X_{\epsilon'}(z_2) = \frac{(q^{-(\epsilon + \epsilon')k/2 + k - 2} z_1 z_2^{-1}; q^{2k})_{\infty}^{\epsilon \epsilon'}}{(q^{-(\epsilon + \epsilon')k/2 + k + 2} z_1 z_2^{-1}; q^{2k})_{\infty}^{\epsilon \epsilon'}} X_{\epsilon'}(z_2) S_{\epsilon}^-(z_1). \quad (5.150)$$

Relations (3.48) and (5.147) imply that the operators  $Z(\epsilon_1, \dots, \epsilon_s | z_1, \dots, z_s)$  commute also with  $U_q(\hat{h})$ .

Let us now derive the first type of the quantum generalized commutation relations in the

$\mathcal{Z}_q$  enveloping algebra, which is valid only if  $\epsilon_r = -\epsilon_{r+1}$ :

$$\begin{aligned}
& \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | z_1, \dots, z_r, z_{r+1}, \dots, z_s) - \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | z_1, \dots, z_{r+1}, z_r, \dots, z_s) \\
&= S_{\epsilon_1}^-(z_1) \dots S_{\epsilon_s}^-(z_s) X_{\epsilon_1}(z_1) \dots X_{\epsilon_{r-1}}(z_{r-1}) [X_{\epsilon_r}(z_r), X_{\epsilon_{r+1}}(z_{r+1})] \\
&\quad \cdot X_{\epsilon_{r+2}}(z_{r+2}) \dots X_{\epsilon_s}(z_s) S_{\epsilon_1}^+(z_1) \dots S_{\epsilon_s}^+(z_s) \\
&= \frac{\epsilon_r}{q - q^{-1}} S_{\epsilon_1}^-(z_1) \dots S_{\epsilon_s}^-(z_s) X_{\epsilon_1}(z_1) \dots X_{\epsilon_{r-1}}(z_{r-1}) \left( \Psi(z_{r+1} q^{\epsilon_r k/2}) \delta(z_r z_{r+1}^{-1} q^{-\epsilon_r k}) \right. \\
&\quad \left. - \Phi(z_r q^{\epsilon_r k/2}) \delta(z_r z_{r+1}^{-1} q^{\epsilon_r k}) \right) X_{\epsilon_{r+2}}(z_{r+2}) \dots X_{\epsilon_s}(z_s) S_{\epsilon_1}^+(z_1) \dots S_{\epsilon_s}^+(z_s) \\
&= \frac{\epsilon_r}{q - q^{-1}} \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1} \dots \epsilon_s | z_1, \dots, \hat{z}_r, z_{r+1}, \dots, z_s) \\
&\quad \cdot \left\{ \Psi_0 \delta(z_r z_{r+1}^{-1} q^{-\epsilon_r k}) q^{2 \sum_{i>r+1} \epsilon_i} \prod_{i>r+1} \left( \frac{1 - q^{-2-(\epsilon_i+\epsilon_r)k/2} z_i z_{r+1}^{-1}}{1 - q^{2-(\epsilon_i+\epsilon_r)k/2} z_i z_{r+1}^{-1} - 1} \right)^{\epsilon_i} \right. \\
&\quad \left. - \Phi_0 \delta(z_r z_{r+1}^{-1} q^{\epsilon_r k}) q^{-2 \sum_{i>r+1} \epsilon_i} \prod_{i<r} \left( \frac{1 - q^{-2-(\epsilon_i+\epsilon_r)k/2} z_{r+1} z_i^{-1} - 1}{q^{2-(\epsilon_i+\epsilon_r)k/2} z_{r+1} z_i^{-1} - 1} \right)^{\epsilon_i} \right\}, \quad \epsilon_r = \epsilon_{r+1},
\end{aligned} \tag{5.151}$$

where we have used (3.50), (3.52), (3.56), and (5.150). Above, we have also introduced the notation

$$\mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1} \dots, \epsilon_s | z_1, \dots, \hat{z}_r, z_{r+1}, \dots, z_s) = S^- X S^+, \tag{5.152}$$

with

$$\begin{aligned}
S^\pm &= S_{\epsilon_1}^\pm(z_1) \dots S_{\epsilon_{r-1}}^\pm(z_{r-1}) S_{\epsilon_{r+2}}^\pm(z_{r+2}) \dots S_{\epsilon_s}^\pm(z_s), \\
X &= X_{\epsilon_1}(z_1) \dots X_{\epsilon_{r-1}}(z_{r-1}) X_{\epsilon_{r+2}}(z_{r+2}) \dots X_{\epsilon_s}(z_s),
\end{aligned} \tag{5.153}$$

where the hat on  $\hat{\epsilon}$  means that the symbol  $\epsilon$  is omitted.

Using the formal power series

$$\begin{aligned}
\left( \frac{1-a}{1-b} \right)^\epsilon &= (1 - a^{(1+\epsilon)/2} b^{(1-\epsilon)/2}) \sum_{n \geq 0} a^{n(1-\epsilon)/2} b^{n(1+\epsilon)/2}, \\
\Pi_{i=1}^s (1 - z_i z^{-1} q^{a_i}) &= \sum_{j_1, \dots, j_s=0,1} (-1)^{\sum_{i=1}^s j_i} q^{\sum_{i=1}^s a_i} z_1^{j_1} \dots z_s^{j_s} z^{-\sum_{i=1}^s j_i}.
\end{aligned} \tag{5.154}$$

we can expand the products in (5.151) as

$$\begin{aligned}
\prod_{i>r+1} \left( \frac{1 - q^{-2-(\epsilon_i+\epsilon_r)k/2} z_i z_{r+1}^{-1}}{1 - q^{2-(\epsilon_i+\epsilon_r)k/2} z_i z_{r+1}^{-1} - 1} \right)^{\epsilon_i} &= \sum_{j_{r+2}, \dots, j_s=0,1} \sum_{m_{r+2} \geq j_{r+2}, \dots, m_s \geq j_s} \\
(-1)^{\sum_{i>r+1} j_i} q^{\sum_{i>r+1} m_i (2\epsilon_i - (\epsilon_i+\epsilon_r)k/2) - 4\epsilon_i j_i} & z_{r+1}^{-\sum_{i>r+1} m_i} z_{r+2}^{m_{r+2}} \dots z_s^{m_s},
\end{aligned} \tag{5.155}$$

and

$$\begin{aligned}
\prod_{i<r} \left( \frac{1 - q^{-2-(\epsilon_i+\epsilon_r)k/2} z_{r+1} z_i^{-1} - 1}{q^{2-(\epsilon_i+\epsilon_r)k/2} z_{r+1} z_i^{-1} - 1} \right)^{\epsilon_i} &= \sum_{j_1, \dots, j_{r-1}=0,1} \sum_{m_1 \geq j_1, \dots, m_{r-1} \geq j_{r-1}} \\
(-1)^{\sum_{i<r} j_i} q^{\sum_{i<r} m_i (2\epsilon_i - (\epsilon_i+\epsilon_r)k/2) - 4\epsilon_i j_i} & z_1^{-m_1} \dots z_{r-1}^{-m_{r-1}} z_{r+1}^{\sum_{i<r} m_i}.
\end{aligned} \tag{5.156}$$

Substituting these expansions in (5.151) and taking into account the formal Laurent expansions

$$\begin{aligned} \mathcal{Z}(\epsilon_1, \dots, \epsilon_s | z_1, \dots, z_s) &= \sum_{n_1, \dots, n_s \in \mathbb{Z}} \mathcal{Z}(\epsilon_1, \dots, \epsilon_s | n_1, \dots, n_s) z_1^{-n_1} \dots z_s^{-n_s}, \\ \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1} \dots \epsilon_s | z_1, \dots, \hat{z}_r, z_{r+1}, \dots, z_s) &= \\ \sum_{n_1, \dots, n_{r-1}, n_{r+2}, \dots, n_s \in \mathbb{Z}} \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, \hat{n}_r, n_{r+1}, \dots, n_s) & z_1^{-n_1} \dots z_{r-1}^{-n_{r-1}} z_{r+2}^{-n_{r+2}} \dots z_s^{-n_s}. \end{aligned} \quad (5.157)$$

we obtain the first quantum generalized commutation relation satisfied by the Laurent modes:

$$\begin{aligned} &\mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_r, n_{r+1}, \dots, n_s) - \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | n_1, \dots, n_{r+1}, n_r, \dots, n_s) \\ &= \frac{\epsilon_r}{q - q^{-1}} \left( \sum_{j_{r+2}, \dots, j_s=0,1} \sum_{m_{r+2} \geq j_{r+2}, \dots, m_s \geq j_s} \delta_{n_r+n_{r+1}, \sum_{i>r+1} m_i} (-1)^{\sum_{i>r+1} j_i} \right. \\ &\quad \cdot q^{n_r \epsilon_r k + \sum_{i>r+1} m_i (2\epsilon_i - (\epsilon_i + \epsilon_r)k/2) + 2\epsilon_i(1-2j_i)} \\ &\quad \cdot \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_{r-1}, \hat{n}_r, n_{r+1}, n_{r+2} + m_{r+2}, \dots, n_s + m_s) \Psi_0 \\ &\quad - \sum_{j_1, \dots, j_{r-1}=0,1} \sum_{m_1 \geq j_1, \dots, m_{r-1} \geq j_{r-1}} \delta_{n_r+n_{r+1}, -\sum_{i<r} m_i} (-1)^{\sum_{i<r} j_i} \\ &\quad \cdot q^{-n_r \epsilon_r k - 2 \sum_{i>r+1} \epsilon_i + \sum_{i<r} m_i (2\epsilon_i - (\epsilon_i + \epsilon_r)k/2) - 4\epsilon_i j_i} \\ &\quad \cdot \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1 + m_1, \dots, n_{r-1} + m_{r-1}, \hat{n}_r, n_{r+1}, n_{r+2}, \dots, n_s) \Phi_0 \Big), \quad \epsilon_r = -\epsilon_{r+1}. \end{aligned} \quad (5.158)$$

We remark that in the above formula the following relations are meant:

$$\begin{aligned} \delta_{n_r+n_{r+1}, \sum_{i>r+1} m_i} &= \delta_{n_r+n_{r+1}, 0}, \quad \text{if } r = 1, \\ \delta_{n_r+n_{r+1}, -\sum_{i<r} m_i} &= \delta_{n_r+n_{r+1}, 0}, \quad \text{if } r = s - 1. \end{aligned} \quad (5.159)$$

The second type of quantum generalized commutation relations in the  $\mathcal{Z}_q$  enveloping algebra is derived in the case  $\epsilon_r = \epsilon_{r+1}$  as follows:

$$\begin{aligned} &(z_r - z_{r+1} q^{2\epsilon_r}) \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | z_1, \dots, z_r, z_{r+1}, \dots, z_s) \\ &- (z_r q^{2\epsilon_r} - z_{r+1}) \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | z_1, \dots, z_{r+1}, z_r, \dots, z_s) \\ &= S_{\epsilon_1}^-(z_1) \dots S_{\epsilon_s}^-(z_s) X_{\epsilon_1}(z_1) \dots X_{\epsilon_{r-1}}(z_{r-1}) \\ &\quad \cdot ((z_r - z_{r+1} q^{2\epsilon_r}) X_{\epsilon_r}(z_r) X_{\epsilon_{r+1}}(z_{r+1}) - (z_r q^{2\epsilon_r} - z_{r+1}) X_{\epsilon_{r+1}}(z_{r+1}) X_{\epsilon_r}(z_r)) \\ &\quad \cdot X_{\epsilon_{r+2}}(z_{r+2}) \dots X_{\epsilon_s}(z_s) S_{\epsilon_1}^+(z_1) \dots S_{\epsilon_s}^+(z_s) = 0, \quad \epsilon_r = \epsilon_{r+1}. \end{aligned} \quad (5.160)$$

Substituting the Laurent expansion (5.157) in this relation we obtain this second quantum

generalized commutation relation satisfied by the Laurent modes

$$\begin{aligned}
& \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_{r-1}, 1 + n_r, n_{r+1}, \dots, n_s) \\
& - q^{2\epsilon_r} \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | n_1, \dots, n_{r-1}, n_{r+1}, 1 + n_r, \dots, n_s) \\
& = q^{2\epsilon_r} \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_{r-1}, n_r, 1 + n_{r+1}, \dots, n_s) \\
& - \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | n_1, \dots, n_{r-1}, 1 + n_{r+1}, n_r, \dots, n_s), \quad \text{if } \epsilon_r = \epsilon_{r+1}.
\end{aligned} \tag{5.161}$$

Although the first quantum generalized commutation relation looks completely different from its classical analogue due to the appearance of various sums over the indices  $j_i$  and  $m_i$  instead of a single sum over a single index in the classical case [12], we have checked that in the limit  $q \rightarrow 1$  this quantum generalized commutation relation does indeed reduce to its classical analogue, which in our notation reads simply as follows:

$$\begin{aligned}
& \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_r, n_{r+1}, \dots, n_s) - \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | n_1, \dots, n_{r+1}, n_r, \dots, n_s) \\
& = (n_r k + 2\epsilon_r(\epsilon_{r+2} + \dots + \epsilon_s) + \epsilon_r h) \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_{r-1}, \hat{n}_r, n_{r+1}, n_{r+2}, \dots, n_s), \\
& \text{if } n_r + n_{r+1} = 0, \quad \epsilon_r = -\epsilon_{r+1}, \\
& = 2\epsilon_r \sum_{i>r+1} \epsilon_i \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_{r-1}, \hat{n}_r, n_{r+1}, n_{r+2}, \dots, n_i + n_r + n_{r+1}, \dots, n_s), \\
& \text{if } n_r + n_{r+1} > 0, \quad \epsilon_r = -\epsilon_{r+1}, \\
& = -2\epsilon_r \sum_{i<r} \epsilon_i \mathcal{Z}(\epsilon_1, \dots, \hat{\epsilon}_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_i + n_r + n_{r+1}, \dots, n_{r-1}, \hat{n}_r, n_{r+1}, n_{r+2}, \dots, n_s), \\
& \text{if } n_r + n_{r+1} < 0 \quad \epsilon_r = -\epsilon_{r+1}.
\end{aligned} \tag{5.162}$$

The classical analogue of the second quantum generalized commutation relation is simpler and given by:

$$\begin{aligned}
& \mathcal{Z}(\epsilon_1, \dots, \epsilon_r, \epsilon_{r+1}, \dots, \epsilon_s | n_1, \dots, n_r, n_{r+1}, \dots, n_s) \\
& - \mathcal{Z}(\epsilon_1, \dots, \epsilon_{r+1}, \epsilon_r, \dots, \epsilon_s | n_1, \dots, n_{r+1}, n_r, \dots, n_s) = 0, \quad \text{if } \epsilon_r = \epsilon_{r+1}.
\end{aligned} \tag{5.163}$$

## 6 Conclusions

In this paper, we have introduced a natural quantum analogue of the  $\mathcal{Z}_q$  algebra with arbitrary level  $k$ . In the special cases  $k = 1$  and  $k = 2$  our  $\mathcal{Z}_q$  algebra simplifies considerably and reduces to the only presently available results in the literature, which become special cases of our general construction. Moreover, as a new example of a  $\mathcal{Z}_q$ -module, that is, with  $k > 2$ , we provide an explicit construction of the basis for a generalized Verma module. One would

like to construct all  $\mathcal{Z}_q$ -modules, and from them all  $U_q(\widehat{sl(2)})$ -modules with arbitrary level  $k$ , and especially the standard ones. We believe that the two types of quantum generalized commutation relations derived in the last section will be useful for the latter purpose. Moreover, one would like to diagonalize the off-critical  $Z_k$  statistical models by the elements of the quantum parafermionic algebra, which is obtained from the quotient of the  $\mathcal{Z}_q$  algebra by its subalgebra  $\mathbb{C}[Q]$ , in the same way that the off-critical Ising model has been diagonalized by the quantum Clifford algebra (the special case  $k = 2$  of the quantum parafermionic algebra) [7].

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